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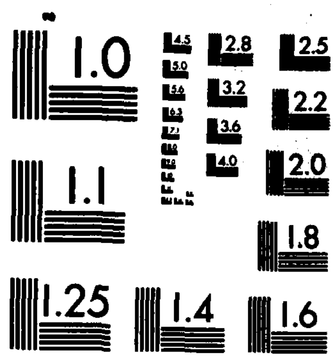

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A UNIFIED METHOD FOR DELAY ANALYSIS OF RANDOM MULTIPLE ACCESS ALGORITHMS

L. Georgiadis, L. Merakos
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Abstract

In this paper, we presents a unified method for the delay analysis of a large class of random multiple-access algorithms. ~~Our~~ This method is based on a powerful theorem referring to regenerative processes, in conjuction with results from the theory of infinite dimensionality linear systems. ~~We~~ The authors apply the method to analyze and compute the per packet expected delays induced by two algorithms, in the presence of the Poisson user model. The considered algorithms are: The controlled ALOHA algorithm, and the "0.487" algorithm. The same method has been previously applied, for the delay analysis of certain limited sensing random access algorithms. Keywords: Communications networks; Allocations,

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1. INTRODUCTION

A key problem in the design of communication networks is the efficient sharing of a common transmission channel, (such as a satellite link, a ground radio channel, a computer bus, a coaxial cable, or an optical fibre) among a large population of network users. This problem is referred to as the multiple-access problem, since many independent users share, and, thus, access a common channel for transmission of information. The solution to the multiple-access problem must incorporate a distributed control scheme, termed multiple-access algorithm, for allocating the channel resources among the network users.

The design and performance of multiple-access algorithms are highly dependent on the nature of the users. When a channel is to support large numbers of bursty (low duty-cycle) users, random multiple-access algorithms (RMAAs) become more efficient than deterministic algorithms. This has been early recognized by the researchers in the field, and a plethora of RMAAs have been proposed during the past fifteen years [1,2].

The key performance measures of a RMAA are its throughput and delay characteristics. The evaluation of such characteristics has been the subject of numerous studies. In most cases, a Markovian model is employed, and the existence of steady state of the random-access system is related to the ergodicity of an underlying Markov process. Depending on the complexity of the state space of such a process, this formulation usually gives sufficient information on the maximum input traffic rate that an algorithm can maintain. However, the evaluation of the delay characteristics is a much harder problem, since they are intimately itnerwoven with the dynamical behavior of the algorithm's scheduling mechanisms. Due to this fact, it is not surprising that results concerning the delay characteristics are limited, and are obtained after a rather intricate and difficult analysis, which is usually matched to the

peculiarities of the specific algorithm at hand.

In this paper, we show how the delay analysis of RMAAs can be unified and simplified, by the use of some known results from the theory of regenerative processes, and the theory of infinite dimensional systems of linear equations. After outlining the method in section 2, we demonstrate its wide applicability and relative simplicity, by applying it, in sections 3 and 4, to two algorithms that represent different classes of RMAAs, namely:

1) the Controlled ALOHA algorithm ("ALOHA-type" class) [6]
2) the "0.487" algorithm ("full sensing-blocked access" class) [7,8]

For the above algorithms, we obtain explicit results on the induced mean delay, for the Poisson infinite-user population model. The higher moments of the delay, for the Poisson as well as for an arbitrary memoryless input stream, can be computed using the same method. We note that the method has been already applied for the delay analysis of a class of limited sensing random access algorithms [11,18,19].

2. THE METHOD

In random-access systems, as in virtually every queueing system, many of the involved stochastic processes are regenerative. The following definition is taken from [4].

Definition The process $\{X_n\}_{n\geq 1}$ is said to be regenerative with respect to the renewal sequence $\{R_i\}_{i\geq 1}$, if for every positive integer M and every sequence $t_1,\ldots,t_M$, with $0<t_1<\ldots<t_M$, the joint distribution of $X_{t_1+R_i},\ldots,X_{t_M+R_i}$ is independent of i. The random variables $R_i, i\geq 1$, and $C_i = R_{i+1} - R_i, i\geq 1$ are referred to as regeneration points and regeneration cycles, respectively.

For regenerative processes, the following elegant and powerful result holds, which will be referred to as the regeneration theorem [3,4,5].

<u>Theorem 1</u>

Let the discrete-time process $\{X_n\}_{n\geq 1}$ be regenerative with respect to the renewal process $\{R_i\}_{i\geq 1}$. Also, let $C_i = R_{i+1} - R_i$, $i=1,2,\ldots$, denote the length of the i-th regeneration cycle, and let f be a nonnegative, real valued, measurable function.

If $C = E\{C_1\} < \infty$ and $S = E\{\sum_{i=1}^{C_1} f(X_i)\} < \infty$, then,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = \lim_{n\to\infty} \frac{1}{n} E\left\{\sum_{i=1}^{n} f(X_i)\right\} = \frac{S}{C}, \qquad \text{w.p. } 1$$

Furthermore, if, in addition to the finiteness of C and S, the distribution of C_1 is not periodic, then X_i converges in distribution to a random variable X_∞, and

$$E\{f(X_\infty)\} = \frac{S}{C}$$

Thus, under the conditions stated above, the limiting (expected) average, and the mean of the limiting distribution of $\{f(X_n)\}_{n\geq 1}$ exist, coincide, and are finite. Moreover, their common value is then given in terms of the per cycle quantities S and C.

Given a RMAA, let $\{X_n\}_{n\geq 1}$ be the process of interest associated with the random-access system; this process might, for example, be the delay process induced by the algorithm. Then, provided that $\{X_n\}_{n\geq 1}$ can be shown to be regenerative, the regeneration theorem itself shows the way to establish the existence of steady state, and to compute the steady-state moments, and the distribution of $\{X_n\}_{n\geq 1}$, by appropriately selecting the function f.

In virtually all existing RMAAs, it is relatively easy to identify regenerative times (e.g., when the system becomes empty, or when an appropriate Markov chain hits a suitable fixed state), at which the process of interest probabilis-

tically restarts itself. Given a RMAA and a function f, the problem then is to exploit the dynamics of the algorithm, to find those per cycle properties of the sample function of the process, that could be subsequently used to evaluate the quantities C and S.

In section 3, it is shown that for the delay process, and for $f(x) = x$, the computation of S and C are intimately related to the solution of an infinite dimensional system of linear equations. It can be shown that this is the case when $f(x) = x^n$, $n = 2,3,\ldots$, as well [11]. Therefore, the steady-state moments of the delay process induced by a particular algorithm, can be computed from the solution of the corresponding infinite linear system. In Appendix A, we give a number of general results, that are useful in establishing the existence and uniqueness of a solution, and in developing approximations to the solution of such systems. In section 4, we apply these results to the specific infinite linear systems developed for the two algorithms of section 3. This procedure involves the following steps.

<u>Step 1</u> Find conditions under which the infinite linear system has a unique, nonnegative solution.

<u>Step 2</u> Show that the variables of interest coincide with the unique solution.

<u>Step 3</u> Develop arbitrarily tight upper and lower bounds on the solution.

3. TWO ALGORITHMS AND THEIR RELATED SYSTEMS OF EQUATIONS

For both algorithms of this section, we assume a collision-type, packet-switched, slotted, broadcast channel. The channel is accessed by a very large (effectively infinite) number of identical, independent, packet-transmitting, bursty users. The cumulative packet generation process is modelled as a Poisson process, with intensity λ packets per slot. However, the proposed method can be applied equally well, when the number of packets per slot are independent and identically distributed (i.i.d) random variables.

We define the delay, $\mathcal{D}_n$, experienced by the n-th arrived packet, as the time difference between its arrival at the transmitter, and the end of its successful

transmission. We are interested in evaluating the steady state statistics of the delay process $\{\mathcal{D}_n\}_{n\geq 1}$, when they exist. Due to space limitations in this paper, we give explicit results, only for the first moment of the delay process. However, higher moments of the delay, as well as other quantities of interest can be computed, using the same method.

3.1 Example 1 : Controlled ALOHA

The earliest and most well known RMAAs belong to the class of the ALOHA techniques [13,6,16]. Here, we analyze a version of the slotted ALOHA algorithm, that operates with each user transmitting a newly arrived packet, in the first slot after its arrival. Should this cause a collision, each involved user independently retransmits its packet in the next slot, with probability f.

A packet whose transmission is unsuccessful is said to be blocked. Let M_i be the number of blocked packets at the beginning of slot i (time segment $[i,i+1)$). This number will be referred to as the backlog size. Also, let R_i denote the number of blocked packets retransmitted in slot i, and N_i denote the number of new packets transmitted in slot i. Given $M_i = m$, then clearly,

$$P(R_i=r) = B_r^m(f) \stackrel{\Delta}{=} \binom{m}{r} f^r (1-f)^{m-r} \ , \ i = 0,1,2,\ldots \qquad (1)$$

$$P(N_i=n) = p_n \stackrel{\Delta}{=} \frac{e^{-\lambda}\lambda^n}{n!} \ , \ i = 0,1,2,\ldots \qquad (2)$$

The delay process induced by the above algorithm "probabilistically restarts itself" at the beginning of each slot T_i, at which $M_{T_i} = 0$, $i = 1,2,\ldots$; this is so because the number of arrivals per slot is an i.i.d. sequence of random variables. Precisely, let $T_1 = 1$, and define T_{i+1} as the first slot after T_i at which $M_{T_{i+1}} = 0$. The interval (T_i, T_{i+1}), $i = 1,2,\ldots$, will be referred to as the ith session.

Let $\mathcal{R}_i$, $i = 1,2,\ldots$, denote the number of packets successfully transmitted in the interval $(0, T_{i+1}]$ (Note that $\mathcal{R}_i$ also represents the number of packets

arrived during the interval $[0, T_{i+1}-1))$. Then, $C_i = R_{i+1} - R_i$, $i = 1,2,\ldots$, is the number of packets successfully transmitted in the interval $(T_i, T_{i+1}]$. The sequence $\{R_i\}_{i\geq 1}$ is a renewal process, since $\{C_i\}_{i\geq 1}$ is a sequence of nonnegative i.i.d. random variables. Furthermore, the delay process $\{\mathcal{D}_n\}_{n\geq 1}$ is regenerative with respect to the renewal process $\{R_i\}_{i\geq 1}$, with regeneration cycle, C_1.

From theorem 1, with $f(\mathcal{D}_i) = \mathcal{D}_i$, we have that if $C = E\{C_1\} < \infty$, and if $S = E\{\sum_{i=1}^{C_1} \mathcal{D}_i\} < \infty$, then, there exists a real number D, such that,

$$D = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathcal{D}_i = \lim_{n\to\infty} \frac{1}{n} E\left(\sum_{i=1}^{n} \mathcal{D}_i\right) = E\{\mathcal{D}_\infty\} = \frac{S}{C} \text{ a.e.}$$

The quantity D will be referred to as the <u>mean packet delay</u>. Next, we develop two systems of equations, whose solution may be used to compute the mean cycle length C, and the mean cumulative delay S. The properties and the computation of the solution will be postponed until section 4.

<u>1.a Mean Cycle Length</u>

If the mean session length $H \stackrel{\Delta}{=} E\{T_{i+1} - T_i\}$, $i \geq 1$, is finite, then by Wald's theorem, we have that,

$$C = \lambda H \tag{3}$$

To determine H, we proceed as follows. Let h_i denote the random number of slots needed to return to zero backlog size, starting from a slot j where the backlog size is equal to i, $i \geq 0$. The operation of the algorithm yields the following relation for the h_i's.

$$h_0 = \begin{cases} 1 & \text{if } N_j = 0, 1 \\ 1 + h_{N_j} & \text{if } N_j > 1 \end{cases} \tag{4.a}$$

$$h_{\substack{i \\ i\geq 0}} = \begin{cases} 1 + h_i & \text{if } R_j + N_j = 0 \\ 1 + h_{i+N_j-1}\ I(i+N_j-1>0) & \text{if } R_j + N_j = 1 \\ 1 + h_{i+N_j} & \text{if } R_j + N_j > 1 \end{cases} \qquad (4.b)$$

where $I(\cdot)$ is the indicator function of the event in the parenthesis.

If we let $H_i = E\{h_i\}$, $i > 0$, then after taking expectations in (4) we obtain,

$$H_i = b_i + \sum_{k=0}^{\infty} c_{ik} H_k\ ,\ i\geq 0 \qquad (5)$$

where $b_i = 1,\ i\geq 0;$

$$c_{00} = c_{01} = 0,\ c_{01} = p_i,\ i>1;\ c_{ik} = p_{k-i},\ k>i+1,\ i\geq 1;$$

$$c_{ii+1} = p_i(1-B_0^i(f)), i\geq 1;\ c_{ii} = p_0(1-B_1^i(f)) + p_1 B_0^i(f),\ i\geq 1;$$

$$c_{10} = 0,\ c_{ii-1} = p_0 B_1^i(f),\ i>1;\ c_{ik} = 0,\ k<i-1,\ i\geq 1;$$

where p_i, $B_j^i(f)$, $i\geq 0$, $0\leq j\leq i$, are as defined in (1), (2), respectively.

Note that the mean session length H, can be computed from system (5), since $H = H_0$.

1.b Mean Cumulative Delay

The mean cumulative delay, S , can be computed using a system of equations similar to system (5). To develop such a system we proceed as follows. Let w_i denote the cumulative delay experienced by all the packets that were successfully transmitted during the h_i slots.(1) Also, let $W_i = E\{w_i\}$, $i \geq 0$, and note that $S = W_0$.

The operation of the algorithm yields the following relations for the w_i's.

$$w_0 = \begin{cases} N_j & \text{if } N_j = 0, \text{ or } 1 \\ N_j + w_{N_j} & \text{if } N_j > 1 \end{cases} \qquad (6.a)$$

---

(1) Here, for convenience, we count the delay of a packet, starting from the beginning of the first slot after its arrival.

$$w_{\substack{i\\i>0}} = \begin{cases} i + N_j + w_i & \text{if } R_j + N_j = 0 \\ i + N_j + w_{i+N_j-1} \quad I(i+N_j-1>0) & \text{if } R_j + N_j = 1 \\ i + N_j + w_{i+N_j} & \text{if } R_j + N_j > 1 \end{cases} \tag{6.b}$$

After taking expectations in (6) we obtain

$$W_i = b_i' + \sum_{k=0}^{\infty} c_{ik} W_k \quad , \; i\geq 0 \tag{7}$$

where $b_0' = \lambda$, $b_i' = i + \lambda$, $i > 1$, and c_{ik} are as defined in (5).

3.2 Example 2 : The "0.487" Algorithm

This algorithm is the most efficient RMAA known to date, for the Poisson infinite-user population model and ternary feedback; (it attains a maximum throughput of 0.487 packets per slot). It is assumed that at the end of each slot i (time segment [i,i+1)), the users receive feedback z_i = 0, 1, or c, if in slot i there were respectively zero, one or more than one packets transmitted. For the description of the algorithm, motivation, and background discussions, the reader is referred to [8], and [15].

Suppose that at the beginning of slot v(time segment (v,v+1]), all packets that arrived before time $t_v < v$, have been successfully transmitted, and there is no information concerning the packets that may have arrived in the interval $[t_v,v)$, (i.e., the distribution of the interarrival times of the packets in $[t_v,v)$ is the same as the one assumed originally). The beginning of such a slot v is called a "collision resolution instant"(CRI). The time difference $d_v = v - t_v$ will be referred to as the "lag at v". In slot v, the users that generated packets in the interval $[t_v, t_v + U_v)$, where $U_v = \min(d_v, \Delta)$, are allowed to transmit; Δ is a parameter to be properly chosen for throughput maximization. In this case, we say that the interval $[t_v, t_v + U_v)$ is "transmitted". After a random number of slots $\mathcal{L}$, and following the rules of the algorithm, another CRI ,v', is reached, with a corresponding $t_{v'}>t_v$. For the analysis of the algorithm, we need the following definitions.

$\delta = t_{v'} - t_v$

N : number of packets in $[t_v, t_{v'})$

ω : sum of delays of the N packets, after the CRI v

ψ : sum of delays of the N packets, until the instant $t_v + U_v$.

$E\{X|u\}$: conditional expectation of the random variable X, given that $U_v = u$

Let $\{v_i\}_{i \geq 1}$ be the sequence of successive collision resolution instants, and let d_i be the lag at v_i. It is known, [9], that the sequence $\{d_i\}_{i \geq 1}$ is a Markov chain, with state space, F a denumerable dense subset of the interval $[1, \infty)$. Let $T_1 = 1$, $d_1 = 1$, and define T_{i+1}, as the first slot after T_i, at which $d_{T_{i+1}} = 1$. From the description of the algorithm it can be seen, after a little thought, that the induced delay process probabilistically restarts itself at the beginning of each slot T_i, $i = 1,2,\ldots$. Therefore, using the notation and definitions of example 1, the mean packet delay D is equal to S/C provided that both S and C are finite.

2.a Mean Cycle Length

As in example 1, if the mean session length $H = E\{T_{i+1} - T_i\}$ is finite, then $C = \lambda H$. To evaluate H we proceed as follows.

Let h_d denote the random number of slots needed to return to lag equal to one, starting from a collision resolution instant v_i with $d_i = d$. Note that, by definition, h_1 is the session length. The operation of the algorithm yields the following relations for the h_d's, $d \in F$.

$$1 \leq d \leq \Delta, \quad h_d = \begin{cases} \ell & \text{if } \ell = 1 \\ \ell + h_{d-\delta+\ell} & \text{if } \ell > 1 \end{cases} \tag{8.a}$$

$$d > \Delta, \quad h_d = \ell + h_{d-\delta+\ell} \tag{8.b}$$

Taking expectations in (8) yields:

$$1 < d < \Delta\ , \quad H_d = E\{\ell|d\} + \sum_{\substack{r,s \\ s\neq 1}} p(r,s|d)H_{d-r+s} \qquad (9.a)$$

$$d > \Delta\ , \quad H_d = E\{\ell|\Delta\} + \sum_{s,r} p(r,s|\Delta)H_{d-r+s} \qquad (9.b)$$

where $p(r,s|x)$ is the joint conditional probability distribution of δ, and ℓ, at the point values r and s, given that the transmitted interval is of length x. Note that,

$$p(r,1|x) = \begin{cases} (1+\lambda x)e^{-\lambda x} & \text{if } r = x \\ 0 & \text{otherwise} \end{cases}$$

System (9) can be written in the form

$$H_d = b_d + \sum_{t\in F} c_{dt}\ H_t\ ,\ d\in F \qquad (10)$$

where $b_d = E\{\ell|d\}$, $1 < d < \Delta$, $b_d = E\{\ell|\Delta\}$, $d > \Delta$, and where c_{dt}, d, t $\in F$ are nonnegative coefficients that can be appropriately identified from (9). The conditional expectation $E\{\ell|d\}$, $1 < d < \Delta$, can be computed as shown in Appendix B.

2.b Mean Cumulative Delay

Let w_d denote the cumulative delay experienced by all the packets that were successfully transmited during the h_d slots. The operation of the algorithm yields the following relation for the w_d's, $d \in F$.

$$1 < d < \Delta\ ,\ w_d = \begin{cases} \omega+\psi & \text{if } \ell = 1 \\ \omega+\psi+w_{d-\delta+\ell} & \text{if } \ell > 1 \end{cases}$$

$$d \geq \Delta\ ,\ w_d = \ \omega+\psi+(d-\Delta)N+w_{d-\delta+\ell} \quad \text{if} \quad \ell \geq 1$$

Taking expectations, we obtain,

$$1 < d < \Delta, \quad W_d = E\{\omega|d\} + E\{\psi|d\} + \sum_{\substack{r,s \\ s\neq 1}} p(s,r|d)W_{d-r+s} \qquad (11.a)$$

$$d > \Delta, \quad W_d = E\{\omega|\Delta\} + E\{\psi|\Delta\} + (d-\Delta)E\{N|\Delta\} + \sum_{s,r} p(s,r|\Delta)W_{d-r+s}$$

$$(11.b)$$

System (11) can be written in the form

$$W_d = b_d' + \sum_{t\in F} c_{dt}\, W_t \ , \ d \in F \qquad (12)$$

where $b_d' = E\{\omega|d\}+E\{\psi|d\}$, $1 < d < \Delta$, $b_d' = E\{\omega|\Delta\}+E\{\psi|\Delta\} + (d-\Delta)E\{N|\Delta\}$, and where the coefficients c_{dt}, d, $t \in F$ are as defined in (10). The conditional expectations $E\{\omega|d\}$, $E\{\psi|d\}$, $1 < d < \Delta$, and $E\{N|\Delta\}$ can be computed as shown in Appendix B.

4. SYSTEM SOLUTION AND MEAN PACKET DELAY BOUNDS

In this section, we investigate the conditions under which the infinite dimensional linear systems (5), (7), (10), and (12) have unique nonnegative solutions, and we develop upper and lower bounds on those solutions. These bounds are then used to obtain bounds on the mean packet delay. We proceed, following the steps outlined in section 2.

4.1 Step 1

For convenience, we rewrite an infinite linear system in an operator form. Specifically, let E be the space of sequences $X = \{x(v)\}: A \to R$, where A is a countable set. Also, let E^L be the subspace of E for which,

$$\sum_{v\in A} |c^L_{\mu v} x(v)| < \infty, \ \mu \in A \ , \ v \in A, \ c^L_{\mu v} \in R$$

We define the operator $L = \{L_\mu(x)\}: E^L \to E$, as follows.

$$L_\mu(x) = b_\mu^L + \sum_{v\in A} c^L_{\mu v} x(v) \ , \ \mu \in A \ , \ x \in E^L \ , \ b^L_\mu \in R$$

In this notation, systems (5), (7), (10), and (12) can be written in the form,

$$S^L = L(S^L),\ S^L \in E^L \tag{13}$$

We are interested in the existence and uniqueness of nonnegative points $S^L \in E^L$, that satisfy (13); such points will be referred to as <u>fixed points</u> of L, and represent solutions to the corresponding infinite linear system of equations. The question of uniqueness of a fixed point S^L, or equivalently of the solution, $\{s^L(i)\}$, to the system that operator L represents, depends upon what conditions are imposed on the solution. Thus, after the existence of a solution, $\{s^L(i)\}$, has been established, one has to indicate a class of sequences in which the solution is unique. If the algorithmic sequences of interest $\{H_i\}$, or $\{W_i\}$ belong to the indicated class, then they must coincide with the solution $\{s^L(i)\}$. (This will be examined in Step 2).

Appendix A includes a number of results that can be used to establish existence and uniqueness of a fixed point of an operator. Depending on the operator, some are more straightforward to apply than others. Among the results in Appendix A that can be used to establish existence of a solution, Lemma A.2 is usually the most useful. According to Lemma A.2, to establish existence of a nonnegative fixed point, S^L, of a nonnegative operator, L, it suffices to find a point $X^o \in E^L$, such that,

$$0 < L(X^o) < X^o \tag{14}$$

A point X^o, satisfying (14), also serves as an upper bound on S^L. Furthermore, to establish a lower bound on S^L, it suffices to find a point $Y^o \in E^L$, such that,

$$Y^o < L(Y^o) < X^o \tag{15}$$

Thus, under (14) and (15), we have that,

$$Y^o < S^L < X^o \tag{16}$$

We proceed now with the analysis of the systems developed in section 3.

1. Controlled ALOHA

System (5) -- Existence: System (5) corresponds to an operator L_1 with $b_\mu^{L_1} = b_\mu$, $c_{\mu\nu}^{L_1} = c_{\mu\nu}$, $\mu, \nu \in N_o$, where N_o is the set of nonnegative integers, and the b_μ's and $c_{\mu\nu}$'s are as defined in (5). If we let $X^o = \{x^o(k)\}$ with $x^o(0)=c_u$, $x^o(k) = \alpha_u k + \beta_u$, $k \geq 1$, then by straightforward manipulations we have that, for this choice of X_o, (14) is satisfied if and only if the following inequalities are satisfied.

$$\lambda < \xi_k(f) \stackrel{\Delta}{=} p_0 B_1^k(f) + p_1 B_0^k(f), \text{ for every } k \geq 1 \tag{17}$$

$$\alpha_u \geq \sup\left\{\frac{1}{\xi_k(f)-\lambda}, k \geq 1\right\} \tag{18}$$

$$\beta_u \geq (1 - \alpha_u (\xi_1(f)-\lambda)) / (p_0 B_1^1(f)) \tag{19.a}$$

$$c_u \geq 1 + \alpha_u (\lambda-p_1) + \beta_u (1-p_0 - p_1) \tag{19.b}$$

It can be readily seen from (17) that if the retransmission probability f is constant in every slot, then there is no $\lambda > 0$ for which (17) is satisfied. If the retransmission probability f, at each slot i, were allowed to depend on the current backlog size, M_i, in accordance to a stationary control policy $f = f(M_i)$, then it is of interest to choose $f(\cdot)$ so that it maximizes the set of λ's for which inequality (17) is satisfied. This is equivalent to maximizing $\xi_k(f)$ with respect to f. It can be easily verified that, for every $k \geq 1$, $\xi_k(f)$ is maximized for $f(k) = f^*(k)$, where [2]

2. We should mention that, in a distributed environment, the backlog size dependent retransmission probability $f^*(\cdot)$ is nonimplementable, since users are not aware of the current backlog size. However, the control policy given by (20) can be implemented approximately by adaptive control schemes that estimate the current backlog size using observable feedback information from the past activity on the channel [6,16].

$$f^*(k) = \frac{1-\lambda}{k-\lambda}\ ,\ k \geq 1 \tag{20}$$

From this point on, we assume that f is chosen as in (20). Under this assumption, inequality (17) is satisfied, provided that,

$$\lambda < \inf \{\xi_k(f^*),\ k \geq 1\} = e^{-1}$$

To satisfy inequalities (18), and (19), we choose,

$$\alpha_u = \sup \left\{ \frac{1}{\xi_k(f^*)-\lambda}\ ,\ k > 1 \right\} = \frac{1}{e^{-1}-\lambda} \tag{21.a}$$

$$\beta_u = e^{\lambda} - \alpha_u\ (1-\lambda\ e^{\lambda})\ ,\ c_u = 1 + \alpha_u\ \lambda\ (1-e^{-\lambda}) + \beta_u\ (1-e^{-\lambda} - \lambda e^{-\lambda}) \tag{21.b}$$

Similarly, it is straightforward to show that if $\lambda < e^{-1}$, then the point Y^o with $y^o(0) = c_\ell$, $y^o(k) = \alpha_\ell k + \beta_\ell$, $k \geq 1$, and

$$\alpha_\ell = (\tfrac{1}{2-\lambda}\ e^{-\lambda} - \lambda\ )^{-1},\ \beta_\ell = e^{\lambda} - \alpha_\ell(1-\lambda e^{\lambda}),\ c_\ell = 1 + \alpha_\ell\lambda\ (1-e^{-\lambda}) + \beta_\ell\ (1-e^{-\lambda}-\lambda e^{-\lambda}) \tag{22}$$

satisfies (15). Thus, from (16) and for $\lambda < e^{-1}$ we have that system (5) has a solution, $S^{L_1} = \{s^{L_1}(k)\}$, such that

$$0 < c_\ell \leq s^{L_1}(0) \leq c_u;\ 0 < \alpha_\ell k + \beta_\ell < s^{L_1}(k) < \alpha_u k + \beta_u\ ,\ k \geq 1 \tag{23}$$

where α_u, β_u, c_u are as given by (21), and α_ℓ β_ℓ, c_ℓ are as given by (22).

<u>System (7) -- Existence</u>: System (7) corresponds to an operator L_2 with $b_\mu^{L_2} = b'_\mu$, $c_{\mu\nu}^{L_2} = c_{\mu\nu}$, μ, $\nu \in N_o$, where the b'_μ's and $c_{\mu\nu}$'s are as defined in (7). Due to the fact that b_k is a linear function of k, and since

$$\sum_{k=0}^{\infty} c_{ik} = 1,\ i \geq 1,$$

it can be easily seen that there is no linear sequence $X^o = \{x^o(k)\}$ satisfying (14). However, given $\lambda < e^{-1}$, it is straightforward to show that we can choose coefficients γ_u, δ_u, ζ_u, d_u, γ_ℓ, δ_ℓ, ζ_ℓ, d_ℓ such that the point X^o with $x^o(0) = d_u$, $x^o(k) = \gamma_u k^2 + \delta_u k + \zeta_u$, $k \geq 1$, and the point $y^o(0) = d_\ell$, $y^o(k) = \gamma_\ell k^2 + \delta_\ell + \zeta_\ell$, $k \geq 1$, satisfy (14), and (15), respectively. The following is such a choice:

$$\gamma_u = 0.5\ (e^{-1}-\lambda)^{-1},\ \delta_u = 2\gamma_u\ (\lambda+\gamma_u(\lambda + \lambda^2 + e^{-\lambda})) \tag{24.a}$$

$$\zeta_u = \zeta(\gamma_u,\delta_u) \triangleq \lambda e^{\lambda} + \gamma_u\ (1 + \lambda\ (1 + \lambda)\ \ e^{\lambda}) + \delta_u\ (\lambda e^{\lambda} - 1) \tag{24.b}$$

$$d_u = d\ (\gamma_u,\ \delta_u,\ \zeta_u) \triangleq \lambda + \lambda\ (1+\lambda - e^{-\lambda})\ \gamma_u + \lambda\ (1-e^{-\lambda})\ \delta_u + (1-e^{-\lambda}-\lambda e^{-\lambda})\zeta_u \tag{24.c}$$

$$\gamma_\ell = 0.5\ (\tfrac{1}{2-\lambda}\ e^{-\lambda}-\lambda)^{-1},\ \delta_\ell = 2\gamma_\ell\ (\lambda + \gamma_\ell(\lambda + \lambda^2 + e^{-1}(1-2\lambda))) \tag{25.a}$$

$$\zeta_\ell = \zeta\ (\gamma_\ell,\ \delta_\ell),\ d_\ell = d\ (\gamma_\ell,\ \delta_\ell,\ \zeta_\ell) \tag{25.b}$$

where $\zeta(\cdot,\cdot)$, $d(\cdot,\cdot,\cdot)$ are as defined in (24.b), (24.c), respectively. Thus, if $\lambda < e^{-1}$, then system (7) has a solution $S^{L_2} = \{s^{L_2}(k)\}$, such that,

$$0 < d_\ell \leq s^{L_2}(0) \leq d_u \tag{26.a}$$

$$0 < \gamma_\ell k^2 + \delta_\ell k + \zeta_\ell < s^{L_2}(k) < \gamma_u k^2 + \delta_u k + \zeta_u\ ,\ k > 0 \tag{26.b}$$

where γ_u, δ_u, ζ_u, d_u are as given by (24), and γ_ℓ, δ_ℓ, ζ_ℓ, d_ℓ are as given by (25).

Systems (5) and (7) -- Uniqueness

We will show that both the solution $\{s^{L_1}(i)\}$ of system (5) and the solution $\{s^{L_2}(i)\}$ of system (7) are unique in the class

$$E_2 = \left\{ X : \sup_{i \in N_o} \frac{|x(i)|}{i^{2+c}} < \infty \right\}$$

where c is a positive constant.

We start with system (7). Since L_2 is majorant of itself, from theorem A.1,

we have that L_2 has a principal fixed point $S_*^{L_2}$, such that $0 < S_*^{L_2} < S^{L_2}$. According to theorem A.2, the fixed point S^{L_2} is unique in the class,

$$E_*^{L_2} = \left\{ X : \sup_{i \in N_o} \frac{|x(i)|}{s_*^{L_2}(i)} < \infty \right\},$$

provided that $Y^o \in E_*^{L_2}$. Since, $Y^o \in E_2$, S^{L_2} will be unique in E_2, if we show that $E_*^{L_2} = E_2$. According to lemma A.1, it suffices to show that,

$$\sup_{i \in N_o} \frac{s_*^{L_2}(i)}{i^2+c} < \infty \tag{27}$$

and

$$\inf_{i \in N_o} \frac{s_*^{L_2}(i)}{i^2+c} > 0 \tag{28}$$

Since $0 < s_*^{L_2}(i) < s^{L_2}(i)$, (27) follows from (26). To show that (28) holds, we use the power sequence, $\{S_n^{L_2}\}_{n \geq 1}$, of L_2 with initial point 0. By definition (see Appendix A), $S_n^{L_2}$ is the point that results after L_2 operates n times on the zero point, (i.e., $S_n^{L_2} = L_2^n(0)$), and $S_n^{L_2} \to S_*^{L_2}$, as $n \to \infty$. Due to the fact that $b_i^{L_2} > 0$, $c_{ik} > 0$, $i, k \in N_o$, we have that $0 < s_n^{L_2}(i) < s_{n+1}^{L_2}(i) < s_*^{L_2}(i)$, for every $n > 1$, $i > 0$. Also, it can be readily shown by induction that, for every $i > 1$, $n > 1$,

$$s_n^{L_2}(i) > ni - n(n-1)/2 \tag{29}$$

From (29) we obtain,

$$\lim_{i\to\infty} \inf \frac{s_*^{L_2}(i)}{i^2+c} \geq \lim_{i\to\infty} \inf \frac{s_i^{L_2}(i)}{i^2+c} \geq \lim_{i\to\infty} \frac{i^2+i}{2(i^2+c)} = \frac{1}{2} \qquad (30)$$

(26) follows from (30), and the fact that $s_*^{L_2}(i) > 0$, $i \geq 0$.

The uniqueness in E_2 of the solution $\{s^{L_1}(i)\}$ of system (5) follows from theorem A.4, part (ii), after one identifies L_1 with O_2 and L_2 with O_1, in the theorem.

2. The "0.487" Algorithm

System (10) -- Existence and Initial Bounds

System (10) corresponds to an operator L_1 with $b_\mu^{L_1} = b_\mu$, $c_{\mu\nu}^{L_1} = c_{\mu\nu}$, $\mu, \nu \in F$, where the b_μ's and $c_{\mu\nu}$'s are as defined in (10). To establish the existence of a nonnegative solution to system (10), we follow the same procedure as in system (5).

Let $X^0 = \{x^0(d)\}$ with $x(d) = \alpha_u d + \beta_u$, $d \in F$, and let $X' = L_1(X^0)$. After straightforward manipulations, we obtain,

$$x'(d) = x^0(d) + E\{\ell|d\} + \alpha_u\left(E\{\ell|d\} - E\{\delta|d\} - (1+\lambda d)e^{-\lambda d}\right) - \beta_u(1+\lambda d)e^{-\lambda d} , \quad 1 \leq d \leq \Delta \qquad (31.a)$$

$$x'(d) = x^0(d) + E\{\ell|d\} - \alpha_u(E\{\delta|\Delta\} - E\{\ell|\Delta\}), \quad d > \Delta \qquad (31.b)$$

According to Lemma A.2, to establish the existence of a nonnegative fixed point of L_1, it suffices to show that there exist α_u, β_u, such that,

$$0 \leq x'(d) \leq x^0(d) \quad , \text{ for every } d \in F \qquad (32)$$

If the condition

$$E\{\delta|\Delta\} > E\{\ell|\Delta\} \tag{33}$$

holds, then it can be readily seen from (31) that (32) is satisfied, if we choose α_u, β_u as follows:

$$\alpha_u = \frac{E\{\ell|\Delta\}}{E\{\delta|\Delta\} - E\{\ell|\Delta\}} \tag{34.a}$$

$$\beta_u = \max\{-\alpha_u, \sup_{1 \le d \le \Delta} (\rho(d))\} \tag{34.b}$$

where

$$\rho(d) = \frac{E\{\ell\ |d\} + \alpha_u(E\{\ell|d\} - E\{\delta|d\} - (1+\lambda d)\exp(-\lambda d))}{(1+\lambda d)\exp(-\lambda d)}$$

The conditional expectations appearing in the above expressions can be computed as shown in Appendix B.

Similarly, it can be shown that, under (33), the point $Y^o = \{y^o(d)\}$ with $y^o(d) = \alpha_\ell d + \beta_\ell$, $d \in F$ satisfies the inequality $Y^o \le L(Y^o) \le X^o$, if α_ℓ and β_ℓ are chosen as follows:

$$\alpha_\ell = \alpha_u\ , \quad \beta_\ell = \inf_{1 \le d \le \Delta} (\ \rho(d)) \tag{35}$$

where α_u, $\rho(d)$ are as given by (34).

Thus, if (33) holds, then from lemma A.2 we have that system (10) has a nonnegative solution S^{L_1}, such that,

$$\alpha_\ell d + \beta_\ell \le s^{L_1}(d) \le \alpha_u d + \beta_u\ , \ d \in F \tag{36}$$

where α_u, β_u, and α_ℓ, β_ℓ are as given by (34) and (35), respectively.

System (12) -- Existence and Initial Bounds

Let L_2 be the operator that corresponds to system (12). Also, let $X^o = \{x^o(d)\}$ with $x^o(d) = \gamma_u d^2 + \delta_u d + \zeta_u$, $d \in F$, and $Y^o = \{y^o(d)\}$ with $y^o(d) = \gamma_\ell d^2 + \delta_\ell d + \zeta_\ell$, $d \in F$.

Following the same procedure as for system (10), we can show that if (33) holds, then system (12) has a nonnegative solution $S^{L_2} = \{s^{L_2}(d)\}$, $d \in F$, such that,

$$\gamma_\ell d^2 + \delta_\ell d + \zeta_\ell \le s^{L_2}(d) \le \gamma_u d^2 + \delta_u d + \zeta_u \qquad (37)$$

where,

$$\gamma_u = \gamma_\ell = \frac{E\{N|\Delta\}}{2(E\{\delta|\Delta\} - E\{\ell|\Delta\})}$$

$$\delta_u = \delta_\ell = \frac{E\{\omega|\Delta\} + E\{\psi|\Delta\} - \Delta\, E\{N|\Delta\} + \gamma_u\, E\{(\delta-\ell)^2|\Delta\}}{E\{\delta|\Delta\} - E\{\ell|\Delta\}}$$

$$\zeta_u = \sup_{1\le d\le\Delta} (\phi(d)) , \quad \zeta_\ell = \inf_{1\le d\le\Delta} (\phi(d))$$

$$\phi(d) = \frac{E\{\omega|d\}+E\{\psi|d\}+\gamma_u(E\{(\delta-\ell)^2|d\}-2dE\{\delta-\ell|d\}-(1+\lambda d)e^{-\lambda\Delta})-\delta_u(E\{\delta-\ell|d\}+(1+\lambda d)e^{-\lambda d})}{(1+\lambda d)e^{-\lambda d}}$$

The conditional expectations in the above expressions can be computed as shown in Appendix B.

Remark It is known,[7],that inequality (33) is satisified if $\lambda < \lambda_m(\Delta)$; where $\lambda_m(\Delta)$ is maximized for $\Delta \cong 2.6$, and $\lambda_m(2.6) \cong 0.4871$.

Systems (10) and (12) -- Uniqueness

We will show that both systems (10) and (12) have unique solutions in the class

$$E_2 = \left\{ X : \sup_{d \in F} \frac{|x(d)|}{d^2} < \infty \right\} \tag{38}$$

As in the case of systems (7) and (9) in example 1, if we show uniqueness for system (12), then the uniqueness for system (10) follows from theorem A.4, part (ii).

According to theorem A.2, the fixed point S^{L_2} is unique in the class

$$E_*^{L_2} = \left\{ X : \sup_{d \in F} \frac{|x(d)|}{s_*^{L_2}(d)} < \infty \right\}$$

provided that $Y^o \in E_*^{L_2}$. Since, by construction, $Y^o \in E_2$, S^{L_2} will be unique in E_2, if $E_*^{L_2} = E_2$. To show that the latter holds, we proceed as follows.

Let $\{S_n^{L_2}\}_{n \geq 1}$ be the power sequence of L_2 with initial point 0. Clearly,

$$s_1^{L_2}(d) = b_d^{L_2} > \epsilon > 0, \text{ for every } d \in F \tag{39}$$

Also, it can be easily shown by induction that,

$$s_n^{L_2}(d) = n((d-\Delta)E\{N|\Delta\}+E\{\omega|\Delta\}+E\{\psi|\Delta\}) - \frac{n(n-1)}{2}(E\{\delta|\Delta\}-E\{\ell|\Delta\})E\{N|\Delta\} \tag{40}$$

for every $d \in F$, $n > 1$, such that $d > n\Delta$. For $d > 2\Delta$, letting[3] $n = \left\lfloor \frac{d}{\Delta} \right\rfloor - 1$ in (40), and using the fact that $\left\lfloor \frac{d}{\Delta} \right\rfloor > \frac{d}{\Delta} - 1$, yields,

$$s_n^{L_2}(d) > \alpha d^2 + \beta d + \gamma \;, \; d > 2\Delta \;, \; n \geq \left\lfloor \frac{d}{\Delta} \right\rfloor - 1 \tag{41}$$

where $\alpha > 0$. (The expressions for the coefficients α, β, γ are not of interest and, therefore, are omitted).

If $S_*^{L_2}$ is the principal solution of L_2, then from lemma A.2 we have,

$$s_*^{L_2}(d) > s_n^{L_2}(d) > 0 \;, \quad \text{for every } d \in F \;, \; n > 1 \tag{42}$$

From (39) and (42) we have that,

$$s_*^{L_2}(d) > \max\,(\varepsilon, \alpha d^2 + \beta d + \gamma) \;, \; \forall\, d \in F \tag{43}$$

From (43) we conclude that,

$$\inf_{d \in F} \frac{s_*^{L_2}(d)}{d^2} > 0 \tag{44}$$

From (37), and the fact that $S_*^{L_2} < S^{L_2}$, we have,

$$\sup_{d \in F} \frac{s_*^{L_2}(d)}{d^2} < \infty \tag{45}$$

Finally, from (44), (45), and lemma A.1 we have that $E_*^{L_2} = E_2$

3. $\lfloor a \rfloor$ denotes the maximum integer not exceeding a.

4.2 Step 2

In step 1, we have established conditions for the existence of nonnegative solutions to the systems of interest, and we have identified classes of sequences in which these solutions are unique. Here, we show that the algorithmic sequences $\{H_i\}$, $\{W_i\}$, where $H_i = E\{h_i\}$ and $W_i = E\{w_i\}$, belong to the corresponding identified class, and therefore, coincide with the unique solution in the class. The proof is based on theorem A.6, and is the same for the two algorithms.

For the case of the sequence $\{H_i\}$, let, in theorem A.6, $L = L_1$, $X_i = h_i$, and $X_i^n = \min(h_i, n)$, $n = 1,2,3,\ldots$. By definition, the X_i's and X_i^n's satisfy condition (a) in the theorem. Condition (b) follows from the fact that $X_i^n \leq n$ a.e. . Finally, condition (c) follows from the operation of the algorithm. Thus, $\{H_i\} = S^{L_1}$.

Similarly, to show that $\{W_i\} = S^{L_2}$, we apply theorem A.6, with $L = L_2$, $X_i = w_i$, and $X_i^n = \min(w_i, n)$, $n = 1,2,3,\ldots$.

4.3 Step 3

In step 1, we have already found upper and lower bounds, X^o and Y^o, respectively, on the solutions to the systems of interest. These bounds can be improved either by computing the power sequences of the corresponding operators with initial points the bounds X^o and Y^o, (lemma A.2), or by solving finite systems of linear equations that are truncations of the original infinite systems, (theorem A.5). Both methods can provide arbitrarily tight upper and lower bounds. We use the first method in the "0.487" algorithm, and the second method in the controlled ALOHA.

1. Controlled ALOHA

For system (5), we apply theorem A.5 with $L = L_1$ and,

$$u^{L_1}(i) = \alpha_u i + \beta_u \qquad , i \in N_o$$

$$\ell^{L_1}(i) = \alpha_\ell i + \beta_\ell \qquad , i \in N_o$$

$$A_j = \{0,1,2\ldots,j\} \quad , j \in N_o$$

;where α_u, β_u, and α_ℓ, β_ℓ are as given by (21) and (22), respectively. Note that, for $j < \infty$, A_j is a finite set and, therefore, all conditions in the theorem are satisfied. Thus, for $\lambda < e^{-1}$,

$$s^{\Phi_j}(i) < H_i = s^{L_1}(i) < s^{\Theta_j}(i) \ , 0 < i < j$$

where $\{s^{\Theta_j}(i)\}_{0<i<j}$ and $\{s^{\Phi_j}(i)\}_{0<i<j}$ are the unique solutions of the (j+1)-dimensional systems (46) and (47), respectively.

$$H_i^u = b_i^{\Theta_j} + \sum_{k=0}^{j} c_{ik}^{L_1} H_k^u \ , i \ 0 < i < j \tag{46}$$

$$H_i^\ell = b_i^{\Phi_j} + \sum_{k=0}^{j} c_{ik}^{L_1} H_i^\ell \quad , i \ 0 < i < j \tag{47}$$

where $b_i^{\Theta_j}$, $b_i^{\Phi_j}$ are as defined in the theorem with $\rho_i = \sigma_i = b_i^{L_1}$, $0 < i < j$. therein. We solved systems (46) and (47) for j = 50. The obtained upper bound H_o^u and lower bound H_o^ℓ on the mean session length H_o, can be found in table 1, for different values of λ, $(\lambda < e^{-1})$. For system (7) we followed the procedure described above with,

$$L = L_2$$

$$u^{L_2}(i) = \gamma_u\, i^2 + \delta_u\, i + \zeta_u \ , i \in N_o$$

$$\ell^{L_2}(i) = \gamma_\ell\, i^2 + \delta_\ell\, i + \zeta_\ell \quad , i \in N_o$$

$$\rho_i = \sigma_i = b_i^{L_2} , i \in \ A_j = \{0,1,2,\ldots,j\} \ , j \in N_o$$

where γ_u, δ_u, ζ_u are as given by (24), and γ_ℓ, δ_ℓ, ζ_ℓ are as given by (25). The obtained bounds W_o^u, W_o^ℓ on the mean cumulative dealy W_o are included in table 1; they were computed using j = 50. From the regeneration theorem and (3) we have[4],

$$D = \frac{W_o}{\lambda H_o} + 0.5 \tag{48}$$

The upper bound $D^u = W_o^u/(\lambda H_o^\ell) + 0.5$, and the lower bound $D^\ell = W_o^\ell/(\lambda H_o^u)+0.5$ on D are included in table 1. Note that, according to theorem A.5, arbitrarily tight bounds can be obtained by increasing j. From a theoretical view point the bounds become exact as $j \to \infty$.

2. The "0.487" Algorithm

From section 4.2 we have that, for $\lambda < 0.487$, $H_d = s^{L_1}(d)$, $d \in F$, and $W_d = s^{L_2}(d)$, $d \in F$, where $\bar{S}^{L_1}$ and S^{L_2} are the fixed points identified in section 4.1. According to lemma A.2 we have that,

$$L_1^n(Y_1^o) < S^{L_1} < L_1^n(X_1^o) \quad , n=1,2,\ldots, d \in F \tag{49}$$

$$L_2^n(Y_2^o) < S^{L_2} < L_2^n(X_2^o) \quad , n=1,2,\ldots, d \in F \tag{50}$$

where $X_1^o = \{\alpha_u d + \beta_u\}_{d \in F}$, $Y_1^o = \{\alpha_\ell d + \beta_\ell\}_{d \in F}$

$$X_2^o = \{\gamma_u d^2 + \delta_u d + \zeta_u\}_{d \in F}, \; Y_2^o = \{\gamma_u d^2 + \delta_u d + \zeta_u\}_{d \in F}$$

and where α_u, β_u, α_ℓ, β_ℓ, γ_ℓ, δ_ℓ, ζ_ℓ, γ_u, δ_u, ζ_u are as given by (34), (35), and (37). For n = 1, and d = 1, (49) yields the following bounds on the mean session length H_1:

$$H_1^\ell < H_1 < H_1^u$$

4. The additional 0.5 units of time represent the mean delay of a packet, until the beginning of the first slot following its arrival. (See footnote 1).

where

$$H_1^u = E\{\ell|1\} + \alpha_u(1-(1+\lambda)e^{-\lambda} + E\{\ell|1\} - E\{\delta|1\}) + \beta_u(1-(1+\lambda)e^{-\lambda})$$

$$H_1^\ell = H_1^u - (\beta_u - \beta_\ell)(1-(1+\lambda)e^{-\lambda})$$

The above bounds can be found in table 2, for different values of λ, ($\lambda < 0.487$). For $n = 1$, and $d = 1$, (50) yields the following bounds on the mean cumulative delay over a session W_1:

$$W_1^\ell < W_1 < W_1^u$$

where

$$W_1^u = E\{\omega|1\} + E\{\psi|1\} + \gamma_u\left(1-(1+\lambda)e^{-\lambda} + E\{(\delta-\ell)^2|1\} - 2E\{\delta-\lambda|1\}\right)$$
$$+ \delta_u(1-(1+\lambda)e^{-\lambda} - E\{\delta-\lambda|1\}) + \zeta_u(1-(1+\lambda)e^{-\lambda})$$

$$W_1^\ell = W_1^u - (\zeta_u - \zeta_\ell)(1-(1+\lambda)e^{-\lambda})$$

The bounds W_1^u and W_1^ℓ are included in table 2. From the regeneration theorem we have $D = W_1/(\lambda H_1)$. The upper bound $D^u = W_1^u/(\lambda H_1^\ell)$, and the lower bound $D^\ell = W_1^\ell/(\lambda H_1^u)$ on the mean packet delay D are included in table 2. The upper bound is plotted in figure 1, together with the same bound for the controlled ALOHA. We note that tighter bounds can be obtained either by evaluating the bounds given by (49) and (50) for higher values of n, or by the method of truncated systems used in the previous example. In both methods, however, we must first compute the conditional probabilities $p(\delta,\ell|x)$ defined in (9), which is a computationally complex task. Note that for the found bounds, (i.e., for $n = 1$ in (49) and (50)), such a computation is not required.

5. CONCLUSIONS AND PRIOR WORK

In this paper we have introduced a method for the delay analysis of RMAAs, in which the induced packet delay process is regenerative, and we have demonstrated its wide applicability by applying it to two specific examples. The method is based on a well known result from the theory of regenerative processes, which relates the asymptotic statistics of such processes to quantities that refer only to one cycle of the process. The per cycle quantities, (e.g., mean cycle length, expectation of the sum of the values of the process over a cycle), are evaluated from the solution of infinite dimensional systems of linear equations.

In applying the method to the two example-algorithms, we have put emphasis on the methodology and rigorous derivations rather than finding short cuts in the analysis of a particular algorithm. In doing so, the essential simplicity of the method may have been obscured. However, to appreciate the simplicity of the method, we note that only by using Lemma A.2, one can obtain with minimal effort:

1) A lower bound on the maximum input rate that an algorithm maintains with finite delay, (i.e., a lower bound on the maximum stable throughput induced by the algorithm). Note that for the two examples of this paper, the found bound coincides with the maximum stable throughput.

2) Optimal algorithmic parameter choices (e.g., the retransmission probability policy in the ALOHA algorithm, and the window size Δ in the "0.487" algorithm).

3) Initial bounds on the mean packet delay, that can be used (if so desired) to form finite linear systems, whose solution can yield arbitrarily tight bounds on the mean packet delay.

The algorithms that served as examples in this paper have been analyzed in a number of studies. From the literature on ALOHA-type algorithms, we mention the work in [6], where the stability properties of the version of the Controlled ALOHA algorithm considered here have been studied, using a Markovian model. The optimal retransmission policy was derived in [6] using Pake's lemma, but the delay analysis problem was not addressed.

The delay characteristics of the "0.487" algorithm have been studied in [9], using a different approach. In contrast to the method in [9], the method proposed here does not require the computation of steady-state probabilities of the under-

lying Markov chain and, therefore, it is computationally simpler. Furthermore, since our approach is based on the asymptotic properties of regenerative processes, it yields stronger convergence results.

λ	H_0^{ℓ}	H_0^u	W_0^{ℓ}	W_0^u	D^{ℓ}	D^u
0.05	1.00426	1.00426	0.05782	0.05782	1.65163	1.65163
0.10	1.01983	1.01983	0.14067	0.14067	1.87936	1.87936
0.15	1.05361	1.05361	0.27541	0.27541	2.24265	2.24265
0.20	1.12017	1.12017	0.53225	0.53225	2.87576	2.87576
0.25	1.25676	1.25676	1.14710	1.14710	4.15097	4.15097
0.30	1.59883	1.59883	3.39345	3.39345	7.57485	7.57485
0.35	3.48032	3.51077	37.04013	39.39037	30.64403	32.83714

Table 1

Delays for the Controlled ALOHA

λ	H_1^{ℓ}	H_1^u	W_1^{ℓ}	W_1^u	D^{ℓ}	D^u
.01	1.00025	1.0003	.015258	.015258	1.5253	1.5255
.05	1.00395	1.00474	.08234	.082346	1.6348	1.6388
.1	1.025	1.030	.18503	.1859	1.796	1.8130
.15	1.060	1.061	.3212	.3245	2.000	2.040
.2	1.1167	1.11367	.5162	.5254	2.270	2.352
.25	1.2069	1.240	.8243	.8468	2.66	2.80
.3	1.356	1.408	1.381	1.434	3.270	3.525
.35	1.627	1.710	2.6088	2.7423	4.358	4.8151
.40	2.2279	2.374	6.6438	6.8603	6.779	7.670
.45	4.487	4.8536	35.012	37.871	16.030	18.754
.47	9.110	9.916	163.698	178.178	35.125	41.613
.48	21.175	23.122	944.35	1031.12	85.086	101.452

Table 2

Delays for the "0.487" Algorithm

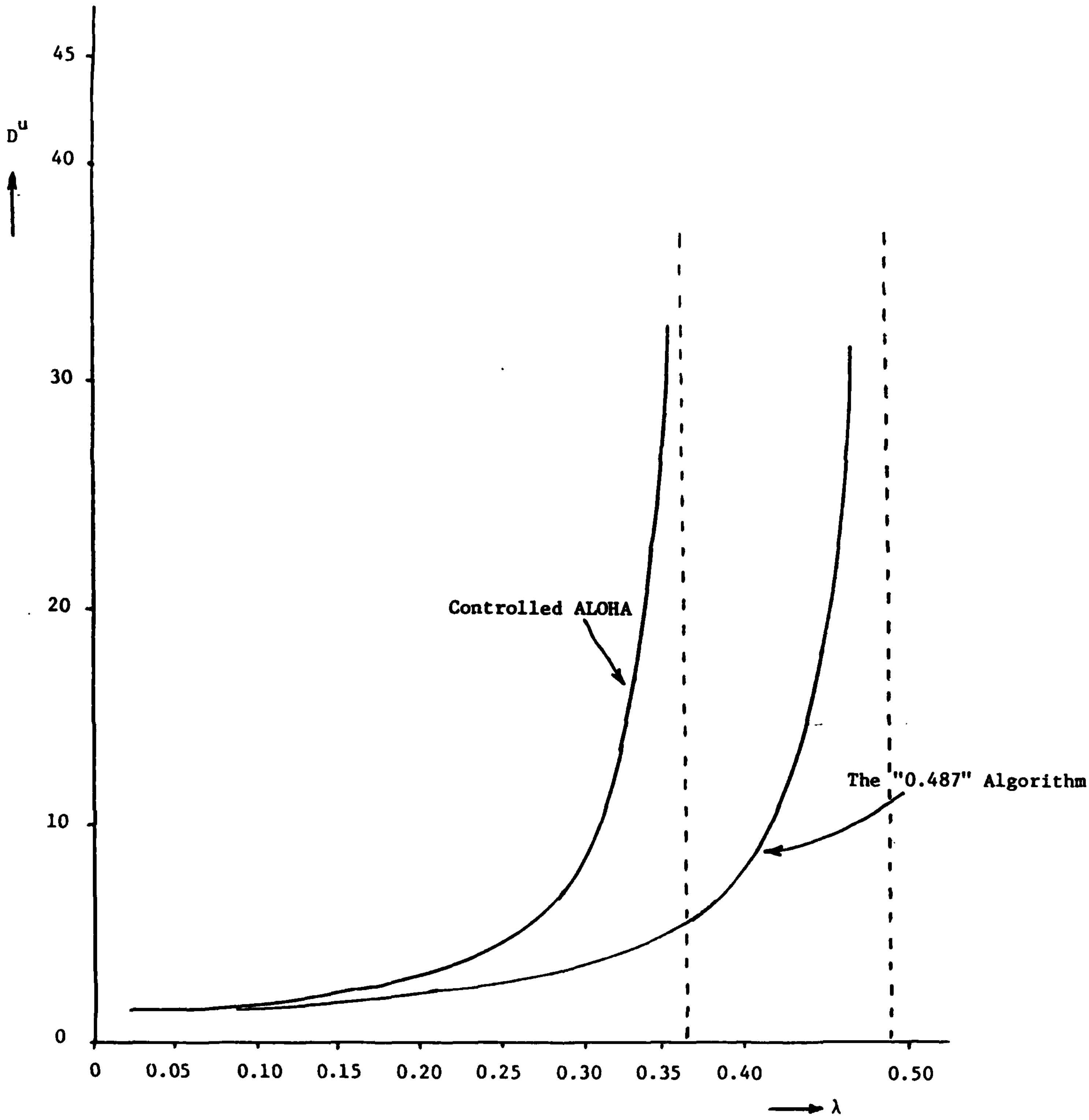

Figure 1

Upper Bounds on Delays--Comparison

APPENDIX A

We present, in a generalized format, some basic results regarding the approximate computation of solutions of infinite dimensionality linear systems [17]. Let A be a denumerable set of indices, and let E be the space of sequences $X = \{x(k)\}: A \to R$. Given a set $\{ c^L_{ik} \varepsilon R,\ b^L_i \varepsilon R, i,k\varepsilon A \}$, let E^L be the subspace of E defined as follows: $E^L = \{X: \sum_{k\varepsilon A} |c^L_{ik}x(k)| < \infty, \forall i \varepsilon A \}$. We define an operator $L : E^L \to E$ as follows: $y(i) = L_i(X) = b^L_i + \sum_{k\varepsilon A} c^L_{ik}x(k), i \ \varepsilon A, X\varepsilon E^L$. A point $S^L \ \varepsilon E^L$, such that,

$$S^L = L(S^L) \tag{A.1}$$

is called a fixed point of the operator L. (A.1) represents an infinite system of linear equations and a fixed point is a solution to this system. Given an operator L, we define its n-th power L^n as follows: $L^1(X_o) \stackrel{\Delta}{=} L(X_o)$, $L^{n+1}(X_o) \stackrel{\Delta}{=} L(L^n(X_o))$, $n=1,2,\ldots$, provided that $X_o \ \varepsilon E^L$, and $L^n(X_o) \ \varepsilon E^L$, for every $n \geq 1$. The sequence $\{x_n\} \stackrel{\Delta}{=} \{L^n(X_o)\}$, $n=1,2,\ldots$ is called the power sequence of L, with initial point X_o. A fixed point of L that is a pointwise limit of the power sequence of L, with initial point $X_o \equiv 0$, is called a principal fixed point of L, and is denoted by S^L_*. An operator $\Theta: E^\Theta \to E$ is called a majorant of L, iff,

$$|c^L_{ik}| \leq c^\Theta_{ik} \qquad i,k \ \varepsilon \ A$$

$$|b^L_i| \leq b^\Theta_i \qquad i \ \varepsilon \ A$$

In this case, L is called a minorant of Θ. The notation $X < X'$, $X \leq X'$, $X,X'\varepsilon E$ means that $x(k) < x'(k)$, $(x(k) \leq x'(k))$, $k \ \varepsilon A$. A point $X \ \varepsilon E$ is called positive (nonnegative) iff, $0 < X$ $(0 \leq X)$. By $|X|$ we denote the sequence defined by

$|x|(k) = |x(k)|$, $k \in A$. Theorems A1, A2 below are essentially theorems I, II § 2 of [17]. They relate the existence and uniqueness of a fixed point of L, to the existence of a fixed point of a majorant Θ of L.

<u>Theorem A.1</u> If Θ is a majorant of L, and Θ has a nonnegative fixed point s^{Θ}, then both Θ and L have principal fixed points s_*^{Θ}, s_*^{L}. Moreover, $0 \leq |s_*^{L}| \leq s_*^{\Theta} \leq s^{\Theta}$

<u>Theorem A.2</u> If Θ is a majorant of L, and Θ has a nonnegative fixed point s^{Θ}, then the principal fixed point s_*^{L} of L is unique in the class $E_*^{\Theta} \subset E^{L}$, defined as follows.

$$E_*^{\Theta} = \{x \in E : \sup_{i \in A} \frac{|x(i)|}{s_*^{\Theta}(i)} < \infty\}^{(1)}$$

Furthermore, s_*^{L} is the pointwise limit of any power sequence of L, with initial point any point in E_*^{Θ}.

Theorem A.3 below relates the existence and uniqueness of a fixed point of L, to the existence of a fixed point of a majorant Θ of L, and it is a consequence of the theory of regular systems [17]. Its difference from theorems A1, A2, lies in the fact that, under the stated assumptions in it, we have, $s^{\Theta} = s_*^{\Theta}$.

<u>Theorem A.3</u> If Θ is a majorant of L, and Θ has a <u>positive</u> fixed point s^{Θ}, such that,

$$\inf_{i \in A} \frac{b_i^{\Theta}}{s^{\Theta}(i)} > 0,$$

then $s_*^{\Theta} = s^{\Theta}$. Therefore, theorem A.2 holds with s_*^{Θ} replaced by s^{Θ}.

(1) We adopt the convention:

$$\frac{0}{0} = 1, \quad \frac{\alpha}{0} = \infty, \quad \alpha > 0$$

The following theorem relates the existence and uniqueness of a fixed point of some operator O_2 to the existence and uniqueness of such a point for another operator O_1, where the latter is not necessarily a majorant of the former.

Theorem A.4 Let O_1, O_2, be two operators such that,

(a) $c_{ik}^{O_1} \geq |c_{ik}^{O_2}| \ \forall\ i,k \in A$, $b_i^{O_1} \in [0,\infty), \forall i \in A$

(i) If O_1 has a fixed point S^{O_1}, and there exists a sequence $g: A \to R$, such that,

(b) $g + S^{O_1} \geq 0$

(c) $\sum_{k \in A} |c_{ik}^{O_1} g(k)| < \infty, \forall i \in A$

(d) $|b_i^{O_2}| \leq (b_i^{O_1} + g(i) - \sum_{k \in A} c_{ik}^{O_1} g(k))\, M,\ \forall i \in A$, for some $M > 0$.

then, O_2 has a fixed point, S^{O_2}.

(ii) If (a), (b), (d) hold, for $g = 0$, then S^{O_2} is unique in the class $E_*^{O_1}$, where $E_*^{O_1}$ is as defined in Th. A.2.

(iii) If in addition to (a), (b), (d), we have that,

(e) $g + S^{O_1} > 0$, and $\inf_{i \in A} \dfrac{b_i^{O_1} + g(i) - \sum_{k \in A} c_{ik}^{O_1} g(k)}{s^{O_1}(i) + g(i)} > 0$,

then the fixed point S^{O_2} of O_2 is unique in the class $E_g^{O_1} \subset E^{O_2}$, defined as follows.

$$E_g^{O_1} = \{x \in E : \sup_{i \in A} \frac{|x(i)|}{s^{O_1}(i) + g(i)} < \infty\}$$

S^{O_2} is the pointwise limit of any power sequence of O_2, with initial point in $E_g^{O_1}$.

Proof

Part (i):Let $Y^{O_1} = (S^{O_1}+g)\,M$. Since $S^{O_1} = O_1(S^{O_1})$, we have that,

$$Y^{O_1}/M - g = O_1\,(Y^{O_1}/M - g)$$

or

$$y^{O_1}(i) = M(b_i^{O_1}+g(i) - \sum_{k\epsilon A} c_{ik}^{O_1} g(k)) + \sum_{k\epsilon A} c_{ik}^{O_1}\, y^{O_1}(i) \qquad (A.2)$$

From (A.2) and (b), we see that the operator Θ with parameters,

$$b_i^{\Theta} = M\,(b_i^{O_1}+g(i)- \sum_{k\epsilon A} c_{ik}^{O_1}\, g(k)),\ i\ \epsilon A$$

$$c^{\Theta}_{ik} = c_{ik}^{O_1} \quad i,k\ \epsilon A\ ,$$

has a nonnegative fixed point $S^{\Theta} = Y^{O_1}$. Because of (α) and (d), Θ is a majorant of O_2. From theorem A.1, we conclude that O_2 has a fixed point.

Part (ii) : This follows from theorem A.2, by observing that $S_*^{\Theta} = MS_*^{O_1}$, and therefore, $E_*^{\Theta} = E_*^{O_1}$.

Part (iii) : Under condition (e), theorem A.3 is applicable, and shows the uniqueness of the fixed point in $E_g^{O_1}$.

The following lemma is useful in identifying the class within which the fixed point of an operator is unique, in the case where the solution of the majorant is not exactly known.

Lemma A.1 If $\{s(i)\}$, $\{\delta(i)\}$: $A \to R$, and,

(α) $\{s(i)\}$, $\{\delta(i)\}$ are nonnegative

(b) $\sup_{i\epsilon A} \dfrac{s(i)}{\delta(i)} < \infty$

(c) $\inf_{i\epsilon A} \dfrac{s(i)}{\delta(i)} > 0$,

then $\sup_{i\epsilon A} \frac{|x(i)|}{\delta(i)} < \infty$, iff $\sup_{i\epsilon A} \frac{|x(i)|}{s(i)} < \infty$, $x \;\epsilon E$, i.e. the classes

$E_S = \{x\epsilon E: \sup_{i\epsilon A} \frac{|x(i)|}{s(i)} < \infty\}$ and $E_F = \{x\epsilon E: \sup_{i\epsilon A} \frac{|x(i)|}{\delta(i)} < \infty\}$ coincide.

<u>Proof</u> For the "if" part let

$$\sup_{i\epsilon A} \frac{|x(i)|}{s(i)} = A < \infty \rightarrow |x(i)| \leq A\, s(i), \quad i \;\epsilon A \tag{A.3}$$

Because of (b), we have,

$$s(i) \leq B\, \delta(i),\ i \;\epsilon A\ ,\ B < \infty \tag{A.4}$$

From (A.3), (A.4), we conclude that $|x(i)| \leq A\, B\, \delta(i)$, $i \;\epsilon A$, or $\sup_{i\epsilon A} \frac{|x(i)|}{\delta(i)} \leq A\, B < \infty$.

The proof of the "only if" part is similar.

The lemma below is used to establish the existence of a fixed point S^L of an operator L, as well as upper and lower bounds on S^L. Monotonicity is proved by induction, while the existence of a fixed point is established via the extended monotone convergence theorem.

<u>Lemma A.2</u> Let L be an operator with nonnegative parameters, i.e. :
$c^L_{ik} \geq 0 \quad i,k \;\epsilon A$, $\qquad b^L_i \geq 0$, $i\epsilon A$. If there exist points Y^o, $X^o \;\epsilon E^L$,
such that,

(a) $Y^o \leq X^o$

(b) $X^o \geq L(X^o) \geq 0$

(c) $Y^o \leq L(Y^o)$,

then the power sequence of L, with initial points X^o (Y^o), decreases (increases) monotonically and pointwise, to a fixed point $S^L(\bar{S}^L)$. Furthermore, $Y^o \leq \bar{S}^L \leq S^L \leq X^o$, and $S^L \geq 0$.

It is generally difficult to establish tight bounds on S^L, using the method exhibited by lemma A.2. The following theorem provides an alternative method for the computation of such bounds. Its proof is based on theorems A.1, A.2, and A.3, and is straightforward.

<u>Theorem A.5</u> Let L be an operator with nonnegative parameters:

$c^L_{ik} \geq 0,\ i,k \in A,$ $\qquad b^L_i \geq 0,\ i \in A.$

Let S^L be a nonnegative fixed point of L, for which it is known that $\ell^L \leq S^L \leq U^L, \ell^L, S^L, U^L \in E^L$. Let $A_j \subset A$, A_j^c be the complement of A_j, and let Φ_j F_j, Θ_j be the operators with parameters,

$$c_{ik}^{\Phi j} = c_{ik}^{\Theta j} = c_{ik}^{F_j} = \begin{cases} c^L_{ik}, & i,k \in A_j \\ 0, & \text{otherwise} \end{cases}$$

$$b_i^{F_j} = \begin{cases} b^L_i + \sum_{k \in A_j^c} c^L_{ik}\, s^L(k), & i \in A_j \\ 0, & \text{otherwise} \end{cases}$$

$$b_i^{\Phi j} = \begin{cases} \rho_i + \sum_{k \in A_j^c} c^L_{ik}\, \ell^L(k), & \rho_i \leq b^L_i,\ i \in A_j \\ 0, & \text{otherwise} \end{cases}$$

$$b_i^{\Theta_j} = \begin{cases} \sigma_i + \sum_{k \in A_j^c} c^L_{ik}\, u^L(k), & \sigma_i \geq b^L_i,\ i \in A_j \\ 0, & \text{otherwise} \end{cases}$$

Then, (a) F_j has a nonnegative fixed point S^{F_j}, such that,

$$s^{F_j}(i) = \begin{cases} s^L(i), & i \in A_j \\ 0, & \text{otherwise} \end{cases}$$

(b) Φ_j is a minorant of F_j, and its principal solution $S_*^{\Phi_j}$ is such that,

$$0 \leq S_*^{\Phi_j} \leq S_*^{F_j} \leq S^{F_j}$$

(c) Θ_j is a majorant of F_j, and if $\sup_{i \in A_j} \frac{b_i^{\Theta_j}}{b_i^{L_j}} < \infty$, then Θ_j has a nonnegative fixed point S^{Θ_j}, such that,

$$0 \leq S_*^{F_j} \leq S_*^{\Theta_j} \leq S^{\Theta_j}$$

(d) If in addition to the previous conditions, also $S^L > 0$, and

$\inf_{i \in A_j} \frac{b_i^L}{s^L(i)} > 0$, then the operators Φ_j, F_j, Θ_j have respective

unique fixed points, S^{Φ_j}, S^{F_j}, S^{Θ_j}, in the class

$$E^{F_j} = \left\{ x \in E : \sup_{i \in A} \frac{|x(i)|}{s^{F_j}(i)} < \infty \right\}, \quad \text{and} \quad S^{\Phi_j} \leq S^{F_j} \leq S^{\Theta_j}$$

<u>Remark</u> If A_j is a finite set with $b_i^L > 0$, $\forall i \in A_j$, the conditions in (c) and (d) are clearly satisfied. If in addition, $\rho_i = \sigma_i = b_i^L$, and $A_j \nearrow A$, then it can be shown that, $S_*^{F_j} \xrightarrow[j\to\infty]{} S^L$, and $S_*^{\Theta_j} \xrightarrow[j\to\infty]{} S^L$, pointwise.

The quantities of interest in the various random access algorithms are statistics of random variables, where many of those statistics are fixed points of some operator L. Theorem A.6 is used to justify the latter statement and appeared in [14].

<u>Theorem A.6</u> Let L be an operator with nonnegative parameters that has a unique nonnegative fixed point S^L in the class $E_g^L = \{x \in E : \sup_{i \in A} \frac{|x_i|}{g(i)} < \infty\}$.

Let $\{x_i^n\}$, $\{x_i\}$, $i \in A$, $n \in N$, be families of random variables, such that,

(a) $0 \leq x_i^n \nearrow x_i$, a.e. for every $i \in A$

(b) $x_i^n \leq M_n g(i)$, a.e. for every $i \in A$, $M_n < \infty$

(c) $f_n \leq L(f_n)$, $f = L(f)$, where $f_n(i) = E\{x_i^n\}$, $f(i) = E\{x_i\}$

Then, f coincides with the unique fixed point S^L in E_g^L.

<u>Proof</u>

We observe that because of (b), $f_n \in E_g^L$, and because of (c) and lemma A.2, $f_n \leq S^L$. From (a) and the monotone convergence theorem, we have that f_n increases to f pointwise; thus, $f \leq S^L$, which implies that $f \in E_g^L$. The assertion now follows from the fact that f is a fixed point of L.

APPENDIX B

In section 4.2, we saw that the computation of conditional expectations, $E\{X|d\}$, is required. In this appendix, we show that those conditional expectations can be computed with high accuracy. Let us define,

$E\{X|d, k\}$: The conditional expectation of the random variable X, given that the arrival interval contains k packets, and has length d.

Then,

$$E\{X|d\} = \sum_{k=0}^{\infty} E\{X|d,k\} e^{-\lambda d} \frac{(\lambda d)^k}{k!} \tag{B.1}$$

Using the rules of the algorithm, the quantities $E\{X|d,k\}$ can be computed recursively , as follows.

$$\begin{aligned}
&E\{\ell/d,k\} = E\{\ell/1,k\};\ \forall\ d \in F \\
&E\{\ell/1,0\} = E\{\ell/1,1\} = 1 \\
&E\{\ell/1,k\} = \Big(1+P_1^k+E\{\ell/1,k-1\}P_1^k+\sum_{i=2}^{k-1} E\{\ell/1,i\}P_i^k\Big)/(1-2P_0^k);\ k \geq 2
\end{aligned} \tag{B.2}$$

where $P_i^k = \binom{k}{i} 2^{-k}$

$$\begin{aligned}
&E\{\delta/d,k\} = d\ E\{\delta/1,k\};\ \forall\ d \in F \\
&E\{\delta/1,0\} = E\{\delta/1,1\} = 1 \\
&E\{\delta/1,k\} = \Big(P_0^k+P_1^k+E\{\delta/1,k-1\}P_1^k+\sum_{i=2}^{k-1} E\{\delta/1,i\}P_i^k\Big)/(2\cdot(1-P_0^k));\ k \geq 2
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
&E\{\ell^2/d,k\}= E\{\ell^2/1,k\};\forall\ d \in F \\
&E\{\ell^2/1,0\}= E\{\ell^2/1,1\} = 1 \\
&E\{\ell^2/1,k\}= \Big(2E\{\ell/1,k\}+2E\{\ell/k,k-1\}\cdot P_1^k+P_1^k-1+P_1^kE\{\ell^2/1,k-1\}+\sum_{i=2}^{k-1} E\{\ell^2/1,i\}P_i^k\Big)/ \\
&\qquad (1-2P_0^k);\ k \geq 2
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
&E\{\delta^2/d,k\}= d^2E\{\delta^2/1,k\}\ ;\ \forall\ d \in F \\
&E\{\delta^2/1,0\}= E\{\delta^2/1,1\} = 1 \\
&E\{\delta^2/1,k\}= \Big(.25(P_1^k+P_0^k)+.5\ P_0^kE\{\delta/1,k\}+.5\ P_1^k\ E\{\delta/1,k-1\}+.25\ E\{\delta^2/1,k-1\}P_1^k+ \\
&\qquad .25 \sum_{i=2}^{k-1} E\{\delta^2/1,i\}P_i^k\Big)/(1-.5P_0^k)\ ;\ k \geq 2
\end{aligned} \tag{B.5}$$

$$E\{\delta\ell/d,k\} = d\ E\{\delta\ell/1,k\}\ ;\ \forall\ d\epsilon F$$

$$E\{\delta\ell/1,0\} = E\{\delta\ell/1,1\} = 1$$

$$E\{\delta\ell/1,k\} = (E\{\delta/1,k\}+.5P_1^kE\{\ell/1,k-1\}+.5P_0^kE\{\ell/1,k\ \ \}+.5\ P_1^k+.5E\{\delta/1,k-1\}P_1^k+ +.5P_1^kE\{\delta\ell/1,k-1\}+.5\sum_{i=2}^{k-1}E\{\delta\ell/1,i\}P_i^k)/(1-P_0^k)\ ;\ k \geq 2 \quad \text{(B.6)}$$

$$E\{N/d,k\} = E\{N/1,k\};\ \forall\ d\epsilon F$$

$$E\{N/1,0\} = 0,\ E\{N/1,1\} = 1$$

$$E\{N/1,k\} = P_1^k+P_1^kE\{N/1,k-1\}+\sum_{i=2}^{k-1}E\{N/1,i\}P_i^k;\ k \geq 2 \quad \text{(B.7)}$$

$$E\{\omega/d,k\} = E\{\omega/1,k\};\ \forall\ d\epsilon F$$

$$E\{\omega/1,0\} = 0,\ E\{\omega/1,1\} = 1 \quad \text{(B.8)}$$

$$E\{\omega/1,k\} = (P_1^k+E\{N/1,k\}+P_1^kE\{N/1,k-1\}+P_1^kE\{\omega/1,k-1\}+\sum_{i=2}^{k-1}E\{\omega/1,k\}P_i^k)/(1-2P_0^k);\ k \geq 2$$

$$E\{\psi/d,k\} = d\ E\{\psi/1,k\};\ \forall\ d\epsilon F$$

$$E\{\psi/1,0\} = 0,\ E\{\psi/1,1\} = \frac{1}{2} \quad \text{(B.9)}$$

$$E\{\psi/1,k\} = (E\{N/1,k\}(1-P_0^k)-P_1^kE\{N/1,k-1\}+.5P_1^k+E\{\psi/1,k-1\}P_1^k+\sum_{i=2}^{k-1}E\{\psi/1,i\}P_i^k) /(2\ (1-P_0^k))\ ;\ k \geq 2$$

From formulas (B.2)-(B.9), we see that a finite number, M, of terms from the infinite series (B.1), can be easily computed. Also, for large k values, and based on the recursive expressions, simple upper and lower bounds on E{X/d,k} can be developed. Those bounds can be used to tightly bound the sum $\sum_{k=M+1}^{\infty} E\{X/d,k\}e^{-\lambda d}\frac{(\lambda d)^k}{k!}$

<u>Remark</u> It can be also proved that

$$E\{N/d\} = \lambda E\{\delta/d\}$$

$$E\{\psi/d\} = \lambda d\ E\{\delta/d\} - \lambda E\{\delta^2/d\}0.5$$

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